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Buckling of Axially Compressed, Core-Filled Cylinders with Transverse Shear Flexibility

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Nomenclature

D_{qx}	= transverse shear rigidity of shell in meridional plane ($= KGh$)
D_x	= shell flexural rigidity in axial direction
E,E_c	= elastic moduli, isotropic shell and core, respectively
E_x,E_y	= elastic moduli of shell in axial and circumferential directions
F_0,F_1	= dimensionless factors defined in Eqs. (21) and (19)
G	= shear modulus of shell material in meridional plane
h	= shell wall thickness
I_0,I_1	= modified Bessel functions of orders zero and one
K	= shear correction factor for shell wall ($=\frac{5}{6}$ for homogeneous shell; $= 1$ for shear-flexible sandwich shell)
k	= core spring constant, defined by Eq. (3) and calculated by Eq. (4)
L	= length of shell
m	= $\pi m_1/L$; m_1 = number of axial half waves in buckle waveform
N	= axial compressive force on shell only
N_{cr}	= critical value of N at which buckling occurs
$\bar{N}_{cr},N_{cr}^*,N_{cr}^0$	= special values of N_{cr}, Eqs. (14–16)
P_{cr}	= critical value of total axial compressive force
p	= normal pressure at core/shell interface due to elastic interaction
Q_0	= amplitude of Q_x
Q_x	= transverse shear stress resultant acting on face cut by position x
R	= radius of shell middle surface; $\bar{R} = mR$
w	= shell displacement normal to its middle surface
w_0	= amplitude of w
x	= axial position coordinate
λ	= $1 - \nu^2$
ν,ν_c	= Poisson's ratios, shell and core materials, respectively
Φ_1	= Seide's core stiffness parameter
Ψ_0	= $I_1(\bar{R})/I_0(\bar{R})$

Introduction

THE problem of axial-compression buckling of circular cylindrical shells containing a soft elastic core is important in the design of solid-propellant rocket-motor cases[1] and of foam-plastic cylinders foamed against a metal mold.[2] Numerous linear, small-deflection analyses of this problem have been performed for thin-walled isotropic cylinders.[3–6] Seide[5] and Yao[6] have shown that the presence of a core results in lower buckling loads for the axisymmetric mode than the unsymmetric ones. As in the case of unfilled cylindrical shells, experimental buckling loads for core-filled cylinders fall much lower than the prediction by linear buckling analysis.[5] However, the discrepancy decreases with increasing values of a dimensionless parameter Φ_1, which is proportional to E_c/E, the core-to-shell elastic modulus ratio. For example, for $\Phi_1 = 0$ (i.e., $E_c = 0$), the ratio of experimental buckling loads to analytical buckling loads averaged ~40%, while at $\Phi_1 = 0.25$, it averaged ~75%. Nonlinear, finite-deflection analyses have been found to serve as a lower bound to the experimental results.[7–8] Yao[6] has concluded that the interfacial shear stresses at the core/shell junction can be neglected, as was done in other analyses of this problem.

Apparently the only buckling analyses of a core-filled orthotropic shell is due to Lemke[9] and Holston.[10] They considered a specially orthotropic shell, but neglected transverse shear flexibility. However, it has been demonstrated that transverse shear flexibility is much more significant in composite-material structures than in homogeneous, isotropic ones of the same dimensions.[11]

The foregoing analyses modeled the core behavior either as a simple Winkler foundation or as a homogeneous, isotropic elastic continuum. Recently Myint[12] treated the problem using Pasternak's two-parameter foundation model, which is equivalent to a Winkler foundation with shear interaction. However, the effect of the second parameter was very weak. (This is consistent with Yao's findings on interfacial shear stress as described previously.)

The purpose of the present Note is to determine the effect of transverse shear flexibility on the buckling loads of core-filled specially orthotropic and isotropic cylinders.

Hypotheses

The following hypotheses form the bases for the analysis:

H1) Displacements are small compared to the shell thickness h and the core radius, so that the strain-displacement relations may be assumed to be linear.

H2) $h/R \ll 1$, where R is the radius of the shell middle surface.

H3) The effect of transverse shear flexibility is included in the same manner as used by Timoshenko[13] for beams and Reissner[14] for plates, i.e., shear distortion is included by use of an appropriate shear factor and transverse normal flexibility is neglected.

H4) The shell is assumed to behave macroscopically as an homogeneous, specially orthotropic, linearly elastic material.

H5) The core is assumed to be represented by an homogeneous, isotropic, linearly elastic continuum, which is related to a Winkler-type foundation by the analysis in Appendix A of Ref. 5.

H6) Only axisymmetric buckling modes are considered, since they are usually predominant in core-filled shell buckling.[5]

H7) Shear stress interaction between the shell and core at their interface is neglected.[6]

H8) Thermal, viscous, and dynamic effects are negligible.

Hypothesis H1 is justified for sufficiently high values of core stiffness. Hypotheses H2 and H3 call for a shell theory analogous to Love's first-approximation theory[15] with transverse shear flexibility added. However, for the axisymmetric case considered here, the Love first-approximation theory is identical to the improved first-approximation theory developed by Sanders.[16] The shell theory used here also is analogous to the Reissner theory of sandwhich shells.[17]

Hypothesis H4 includes isotropic materials as a special case and is approximately applicable to laminated composite

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materials filament wound with alternating angles of $+\theta$ and $-\theta$. The specially orthotropic approximation is good for many layers, but not as good for a few layers, because then bending-stretching coupling can be significant, as demonstrated by Tasi,[18] for example.

The core isotropic behavior in H5 is justified for most solid propellants (but not for filament-reinforced propellants[19] because they are orthotropic) and many foamed plastics (but not thermoplastics[20]). Hypothesis H6 is consistent with the incorporation of transverse shear flexibility, which is important only in thicker shells, which in turn buckle axisymmetrically usually.

Analysis

The linear differential equations governing the axisymmetric buckling of an axially compressed, specially orthotropic cylindrical shell with an isotropic core can be written in the following form:

$$D_x w^{IV} + N w^{11} + (E_y h/R^2) w - (D_x/D_{qx}) Q_x^{111} + p = 0 \quad (1)$$

$$D_x w^{111} - (D_x/D_{qx}) Q_x^{11} + Q_x = 0 \quad (2)$$

where primes denote differentiation with respect to x.

Seide[5] has shown that normal pressure at the interface between the shell and the isotropic core can be represented as follows:

$$p = kw \quad (3)$$

where k is a spring constant depending upon the core properties and dimensions as follows, for the case of axisymmetric deformation:

$$k = [E_c/2(1 - \nu_c^2)R][(\Psi_0^{-2} - 1)\bar{R}^2 - 2(1 - \nu_c)] \quad (4)$$

For very long cylinders

$$k \approx E_c/R(1 - \nu_c) \quad (4a)$$

To satisfy simple-support boundary conditions at both ends on the shell, the following form of single-mode solution is assumed:

$$w = w_0 \sin mx \qquad Q_x = Q_0 \cos mx \quad (5)$$

Substituting Eqs. (3–5) into Eqs. (1) and (2) yields the following coupled linear algebraic equations:

$$(D_x m^4 - N m^2 + E_y h R^{-2} + k) w_0 - (D_x/D_{qx}) m^3 Q_0 = 0 \quad (6)$$

$$-m^3 D_x w_0 + [1 + (D_x/D_{qx})] m^2 Q_0 = 0 \quad (7)$$

Combining Eqs. (6) and (7), taking care not to escalate the order of the combined differential equation to avoid mathematical error, one obtains the following result:

$$N = [(D_x m^2)^{-1} + D_{qx}^{-1}]^{-1} + (E_y h R^{-2} + k) m^{-2} \quad (8)$$

The critical buckling load N_{cr} is the minimum value of N corresponding to a non-negative integer value of m_1, where $m_1 = mL/\pi$.

The contributions of the various factors are readily apparent in the right-hand side of Eq. (8). The $D_x m^2$ factor is the contribution of axial flexural rigidity and the $E_y h R^{-2} m^{-2}$ term is the effect of circumferential extensional rigidity; both of these effects are present in classical theory (thin-walled, unfilled cylinders). The D_{qx}^{-1} factor is the contribution of transverse shear flexibility and km^{-2} is the effect of the core stiffness.

In the general case, Eq. (8) must be evaluated for a number of integer values of m_1 and the lowest value of N selected to be N_{cr}. For the special case of very long cylinders, m_1 is large and thus can be considered to vary continuously rather than in discrete jumps. Under these conditions, N_{cr} can be determined approximately by setting $dN/dm = 0$, to determine m_{cr} and thus N_{cr}. The results, for long cylinders only, are

$$m_{cr} \approx [D_x^{1/2}(E_y h R^{-2} + k)^{-1/2} - (D_x/D_{qx})]^{-1/2} \quad (9)$$

$$N_{cr} \approx 2[(E_y h R^{-2} + k) D_x]^{1/2} - (E_y h R^{-2} + k)(D_x/D_{qx}) \quad (10)$$

Evaluation of Effect of Transverse Shear Flexibility

In Ref. 21, Guz' determined criteria for the applicability of the Kirchhoff-Love hypothesis (i.e., neglect of transverse shear flexibility) in linear analysis of buckling in thin plates. The criterion he used was that N_{cr} be positive; application of this same criterion to Eq. (10) gives

$$2 - D_{qx}^{-1}[(E_y h R^{-2} + k) D_x]^{1/2} > 0 \quad (11)$$

Equation (11) represents the extreme limit of applicability of neglect of D_{qx}^{-1}. Actually the significance of transverse shear flexibility depends upon the percentage error which is acceptable. Neglecting transverse shear only in Eq. (10) yields

$$\tilde{N}_{cr} = 2[(E_y h R^{-2} + k) D_x]^{1/2} \quad (12)$$

Neglecting the core stiffness k only in Eq. (10) gives

$$N_{cr}^* = 2(E_y D_x h/R^2)^{1/2} - (E_y D_x h/R^2 D_{qx}) \quad (13)$$

Finally, neglecting both D_{qx}^{-1} and k results in the classical expression

$$N_{cr}^0 = 2(E_y D_x h/R^2)^{1/2} \quad (14)$$

Defining fractional error as $[(N_{cr})_{approx}/(N_{cr})_{exact}]^{-1}$, the following expressions are obtained: for core-filled shells

$$(\tilde{N}_{cr}/N_{cr}) - 1 = F_1/(1 - F_1) \quad (15)$$

where

$$F_1 \equiv (2D_{qx})^{-1}[D_x(E_y h R^{-2} + k)]^{1/2} \quad (16)$$

For unfilled shells

$$(N_{cr}^0/N_{cr}^*) - 1 = F_0/(1 - F_0) \quad (17)$$

where

$$F_0 = (2D_{qx})^{-1}(D_x E_y h R^{-2})^{1/2} \quad (18)$$

Inspection of Eqs. (15–18) clearly demonstrates that the effect of transverse shear flexibility is greater in long core-filled shells than in unfilled ones. This same conclusion carries over to short shells as well.

Table 1 Buckling load calculations

Theory	Eq.	$(m_1)_{cr}$	N_{cr}/h, psi	Ratio
Classical, no core, $L^{-1} = 0$	16	0.68	7890	0.554
Classical, no core, integer m_1	9[a]	2	7970	0.560
Shear flexible, no core, $L^{-1} = 0$	15	2.54	6730	0.472
Shear flexible, no core, integer m_1	9[b]	3	6905	0.485
Classical with core,	Ref. 5	2.82	18,470	1.295
$L^{-1} = 0$	14	2.92	14,900	1.046
Classical with core, integer m_1	9[c]	3	18,660	1.309
Shear flexible with core, $L^{-1} = 0$	12	3.06	14,280	1.001
Shear flexible with core, integer m_1	9[d]	4	14,260	1.000

[a] Set $k = 0$, $D_{qx}^{-1} = 0$, minimize N_{cr} for integer m_1.
[b] Set $k = 0$, minimize N_{cr} for integer m_1.
[c] Set $D_{qx}^{-1} = 0$, minimize N_{cr} for integer m_1.
[d] Minimize N_{cr} for integer m_1.

Numerical Example

As a specific example, a foamed-in-plane polyurethane cylinder is considered. The core and skin are both assumed to be isotropic and linearly elastic, with the same properties in tension and compression. Using $E_c = 22{,}000$ psi, $\nu_c = \frac{1}{3}$, $E = 42{,}000$ psi, $\nu = \frac{1}{3}$, $h = 0.15$ in., $L = 1.00$ in., and $R = 0.488$ in., the results obtained are as shown in Table 1.

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Electron Energy Distributions in an Ion Engine Discharge

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THE presence of a high-energy, primary electron component in the discharge plasma of mercury electron-bombardment ion engines is well known.[1–4] These primaries, arising from electrons which are accelerated into the plasma through the cathode sheath, have also been found in the present tests in the discharges of laboratory models of ion engines using argon as the working gas. This Note presents the results of energy distribution analyses carried out on the probe curves obtained in such an engine.

Energy Distribution Analysis

The Langmuir probe curves obtained in this type of discharge have the general shape shown in the uncorrected curve of Fig. 1 when plotted semilogarithmically. They are usually analyzed by assuming that the energy distribution of the electrons collected by the probe is composed of a high-energy, monoenergetic component superimposed upon a low-energy, Maxwellian component. This assumption enables the two components to be separated and values for electron densities and energies can then be obtained from the curves. The methods of Strickfaden and Geiler[1] and Knauer et al.[3] lead to slightly different values of the primary electron energy; here the latter method has been employed in analyzing the linear curves, and produces corrected plots of the form shown in Fig. 1. The validity of this method of analysis can be checked by determining the energy distributions found in practice. The usually quoted way of doing this is the Druyvesteyn method,[5,6] which involves taking second derivatives of the probe curve. If this method is carried out graphically, it is inaccurate and tedious.

Medicus[7,8] has developed a simple, quick, accurate graphical method, which has been used here to determine the experimental energy distributions given by Langmuir probe

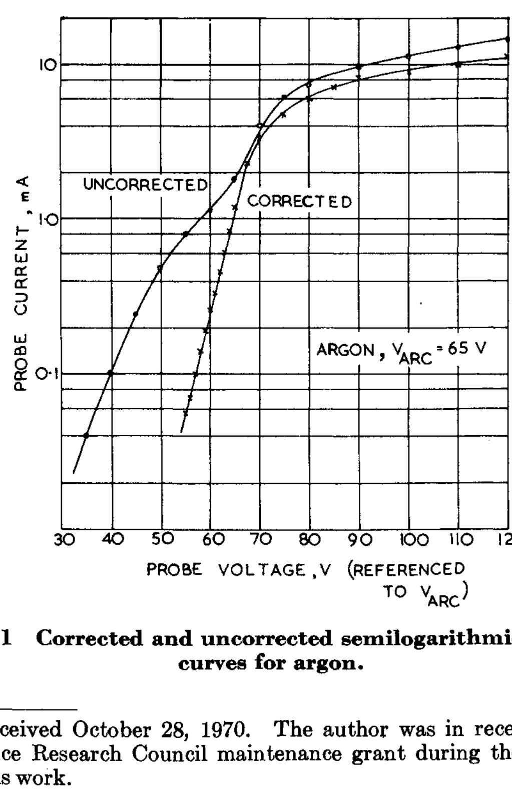

Fig. 1 Corrected and uncorrected semilogarithmic probe curves for argon.

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